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Hamiltonian formulation of exactly solvable models and their physical vacuum states

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ABSTRACT

We study simple two-dimensional models with massless and massive fermions in the Hamiltonian framework. While our ultimate goal is to gain a deeper insight to structural differences between the usual ("spacelike" – SL) and light-front (LF) forms of the relativistic dynamics, an attempt is also made to clarify a few conceptual problems of quantum field theory. We point out that contrary to the assumption of canonical quantization, interacting Heisenberg fields do not always reduce to free fields at $t = 0$. We also show that by incorporating operator solutions of the field equations to the canonical formalism, SL and LF Hamiltonians of the derivative-coupling model as well as of the Federbush model acquire an equivalent structure. In the usual canonical treatment, physical predictions in the two schemes disagree – the SL Hamiltonians contain interaction terms while their LF counterparts do not. Using a Bogoliubov transformation, the physical vacuum of the Thirring model is then derived for the first time. It has a form of a coherent state quadratic in composite boson operators which, after bosonization of the vector current, are present in the (nondiagonal) interaction Hamiltonian. To find the vacuum of the Federbush model by an analogous Bogoliubov transformation, we propose a massive version of Klaiber's current bosonization and demonstrate advantages of the LF treatment of the model.

1. Introduction

The usual "spacelike" (SL) and the light-front (LF) [1] forms of relativistic quantum field theory (QFT) are two independent representations of the same physical reality. There are however striking differences between both schemes already at the level of basic properties [2,3]. This concerns the mathematical structure as well as some physical aspects (nature of field variables, division of the Poincaré generators into the kinematical and dynamical sets, status of the vacuum state, etc.) Exactly solvable models offer an opportunity to study the structure of the two theoretical frameworks and their relationship since in these models exact operator solutions of field equations are known. From the solutions, the correlation functions can be computed nonperturbatively and independently of more sophisticated conformal QFT methods [4]. Note that not all solvable models belong to the conformal class. Thus investigations of their properties in a Hamiltonian approach is a very useful alternative. It permits us to study directly the role of the vacuum state and of the operator structures in both forms of QFT. Let us recall in this connection that in the LF form of the relativistic dynamics, Fock vacuum is often the lowest-energy eigenstate of the *full* Hamiltonian. This unique feature is not present in the SL theory and the (unknown) true vacuum state is in practice often replaced by the lowest-energy eigenstate of the free Hamiltonian (perturbative vacuum) without a deeper justification.

In the present Letter, we give a brief survey of a study, based on the above ideas, of the derivative-coupling model (DCM) [5], the Thirring (TM) [6] and the Federbush model (FM) [7]. All these models are quantum field theories in one space dimension. The unifying idea is to benefit from the knowledge of operator solutions of the field equations to re-express the corresponding SL and LF Hamiltonians purely in terms of true degrees of freedom, namely the free fields. This previously overlooked aspect not only simplifies the overall physical picture but also removes structural differences between SL and LF Hamiltonians. For example, in the case of the simplest theory, the DC model, the conventional canonical procedure applied to the SL and LF Lagrangians leads to a striking result: the SL Hamiltonian contains an interaction term while its LF analog does not. On the other hand, if we modify this procedure as suggested above, the discrepancy disappears:

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the SL version of the DCM Hamiltonian is found to also have the interaction-free form. Consequently, the physical SL vacuum of this extremely simple model coincides with the Fock vacuum in a full agreement with the LF result. However, for the models with more complicated interaction structure, the Fock vacuum is an eigenstate only of the free part of the SL Hamiltonians. This is because the interaction parts of the SL Hamiltonians are generally nondiagonal when expressed in terms of creation and annihilation operators. To find the true vacuum state, they have to be diagonalized. This is a complicated dynamical problem which however turns out to be tractable analytically for the Thirring and Federbush models. Our idea is to bring their Hamiltonians to a quadratic form by bosonization of the vector current and to diagonalize them by a Bogoliubov transformation, generating thereby the true ground state as a transformed Fock vacuum (a coherent state). We will show this explicitly for the Thirring model. As for the Federbush model, the conventional procedure yields a vanishing interaction Hamiltonian for the LF case and a nonvanishing one for the SL case. Although this discrepancy is removed when the solutions of the field equations are taken into account, leading to interaction Hamiltonians of the same structure, the LF scheme maintains clear computational advantages with its much simpler operator structures and with the Fock vacuum being its physical vacuum state. We will discuss the Federbush model only very briefly in the present Letter leaving a more detailed treatment for a subsequent publication [8].

On a more formal level, the knowledge of the explicit form of the operator solutions in the studied models tells us that the interacting Heisenberg field does not reduce to a free field at $t = 0$, contrary to the assumption of canonical quantization. This may have consequences for more complicated models. Finally, the solvability of the (conformally-noninvariant) massive Federbush model allows us to test the methods of conformal field theory where the mass term is treated as a perturbation [4].

2. The derivative-coupling model

It is instructive to explain our main ideas within a very simple theory – massive fermion and scalar fields interacting via a gradient coupling. Its classical Lagrangian and field equations are

$$\mathcal{L} = \overline{\Psi}\left(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - m\right)\Psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2 - g\partial_\mu\phi J^\mu, \tag{1}$$

$$i\gamma^\mu\partial_\mu\Psi = m\Psi + g\partial_\mu\phi\gamma^\mu\Psi, \tag{2}$$

$$\partial_\mu\partial^\mu\phi + \mu^2\phi = g\partial_\mu J^\mu. \tag{3}$$

The original Schroer's model [5] had $\mu = 0$. Our convention for the gamma matrices is $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^2$, $\alpha^1 = \gamma^5 = \gamma^0\gamma^1$ and σ^i are the Pauli matrices. $J^\mu(x)$ is the vector current composed from the interacting fermion fields, $J^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x)$. Classically, the vector current is conserved, $\partial_\mu J^\mu = 0$, and the scalar field satisfies the free Klein–Gordon equation. This feature is not guaranteed to persist on the quantum level. Since Eq. (2) can be solved exactly irrespectively of whether the scalar field $\phi(x)$ is free or interacting, the most natural way of solving the coupled equations (2) and (3) is to use this solution in the correctly defined (regularized) quantum current that will be inserted to the right-hand side of (3). More specifically, the (classical) solution of the Dirac equation (2) is

$$\Psi(x) = e^{-ig\phi(x)}\psi(x), \qquad i\gamma^\mu\partial_\mu\psi(x) = m\psi(x). \tag{4}$$

In quantum theory, the Fock decomposition of the free massive fermion field $\psi(x)$ has the form

$$\psi(x) = \int_{-\infty}^{+\infty} d\tilde{p}^1\left[b(p^1)u(p^1)e^{-i\hat{p}\cdot x} + d^\dagger(p^1)v(p^1)e^{i\hat{p}\cdot x}\right]. \tag{5}$$

It contains the spinors $u^\dagger(p^1) = (\sqrt{p^-}, \sqrt{p^+})$, $v^\dagger(p^1) = (-\sqrt{p^-}, \sqrt{p^+})$, where $p^\pm = E(p^1) \pm p^1$, $E(p^1) = \sqrt{p_1^2 + m^2}$. In the expansion (5), $\hat{p}\cdot x = E(p^1)t - p^1x^1$ and we have used the abbreviation $d\tilde{p}^1 \equiv dp^1/\sqrt{4\pi E(p^1)}$. The fermion and antifermion Fock operators satisfy the anticommutation relations

$$\{b(p^1), b^\dagger(q^1)\} = \{d(p^1), d^\dagger(q^1)\} = \delta(p^1 - q^1). \tag{6}$$

Similarly, the free scalar field, quantized by $[a(k^1), a^\dagger(l^1)] = \delta(k^1 - l^1)$, will be expanded as

$$\begin{aligned}\phi(x) &= \int_{-\infty}^{+\infty} d\tilde{k}^1\left[a(k^1)e^{-i\hat{k}\cdot x} + a^\dagger(k^1)e^{i\hat{k}\cdot x}\right]\\ &\equiv \phi^{(+)}(x) + \phi^{(-)}(x).\end{aligned} \tag{7}$$

The quantum version of the above Lagrangian contains operators whose products are singular if their space–time arguments coincide. A convenient regularization is to separate these arguments by a small amount ϵ (the "point-splitting"). In quantum theory, the solution $\Psi(x)$ (4) has to be regularized, too. A consistent way to do that is to normal-order the exponential in this solution:

$$\Psi(x) = Z^{1/2}(\epsilon)e^{-ig\phi^{(-)}(x)}e^{-ig\phi^{(+)}(x)}\psi(x), \tag{8}$$

where $Z(\epsilon)\exp\{g^2[\phi^{(+)}(x+\epsilon/2), \phi^{(-)}(x-\epsilon/2)]\} = \exp\{-ig^2D^{(+)}(\epsilon)\}$ and $D^{(+)}(x-y)$ is the corresponding two-point function. Applying the point-splitting regularization to the interacting current, we find

$$\begin{aligned}J^\mu(x) &= s\lim_{\epsilon\to 0}\frac{1}{2}\left\{Z(\epsilon)\overline{\psi}\left(x+\frac{\epsilon}{2}\right)e^{ig\phi^{(-)}(x+\frac{\epsilon}{2})}e^{ig\phi^{(+)}(x+\frac{\epsilon}{2})}\right.\\ &\quad\left.\times\,\gamma^\mu e^{-ig\phi^{(-)}(x-\frac{\epsilon}{2})}e^{-ig\phi^{(+)}(x-\frac{\epsilon}{2})}\psi\left(x-\frac{\epsilon}{2}\right) + \text{H.c.}\right\}\\ &= {:}\overline{\psi}(x)\gamma^\mu\psi(x){:} + \frac{g}{2\pi}\partial^\mu\phi(x).\end{aligned} \tag{9}$$

Here $s\lim$ designates the symmetric limit, H.c. means Hermite conjugate and we have used the free-field relation $\overline{\psi}(x+\epsilon/2)\gamma^\mu\psi(x-\epsilon/2) = {:}\overline{\psi}(x)\gamma^\mu\psi(x){:} - \frac{i}{\pi}\frac{\epsilon^\mu}{\epsilon^2}$. Note that all singular terms have been automatically canceled in (9) due to the manifestly hermitian definition of the current, so that no vacuum subtractions are needed. The constant $Z(\epsilon)$ got canceled by the factor $Z^{-1}(\epsilon)$ coming from normal ordering of the two exponentials sandwiching γ^μ in (9). The quantum current $J^\mu(x)$ is not conserved (it is "anomalous"), $\partial_\mu J^\mu(x) = \frac{g}{2\pi}\Box\phi(x)$. However, it is obvious that the only effect of the anomaly is to renormalize the scalar field mass,

$$\partial_\mu\partial^\mu\phi + \tilde{\mu}^2\phi = 0, \quad \tilde{\mu}^2 = \frac{\mu^2}{1-\frac{g^2}{2\pi}}. \tag{10}$$

An analogous calculation of the quantum axial vector current yields

$$J_5^\mu(x) = {:}\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x){:} - \frac{g}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi(x), \tag{11}$$

which is a conserved quantity (due to the conservation of its free part and the presence of $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$).

The conjugate momenta $\Pi_\phi = \partial_0\phi(x) - gJ^0$, $\Pi_\Psi = \frac{i}{2}\Psi^\dagger$, $\Pi_{\Psi^\dagger} = -\frac{i}{2}\Psi$ lead from the Lagrangian (1) to the Hamiltonian $H = H_{0B} + H'$. H_{0B} corresponds to the free massive scalar field and

$$H' = \int_{-\infty}^{+\infty} dx^1 \left[-i\Psi^\dagger\alpha^1\partial_1\Psi + m\Psi^\dagger\gamma^0\Psi\right] + g\int_{-\infty}^{+\infty} dx^1\,\partial_1\phi J^1. \quad (12)$$

Since the term $(i/2)\overline{\Psi}\gamma^\mu\overset{\leftrightarrow}{\partial}_\mu\Psi$ in the Lagrangian is conventionally taken in terms of the free field, the first term in H' becomes simply $-i\psi^\dagger\alpha^1\partial_1\psi$. Setting $m=0$ for simplicity, the interaction Hamiltonian H_g (the last term in Eq. (12)) acquires the form

$$H_g = -\frac{g}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{dk^1\,|k^1|^2}{\sqrt{E(k^1)|k^1|}}\big[a^\dagger(k^1)c(k^1) + c^\dagger(k^1)a(k^1) + a^\dagger(k^1)c^\dagger(-k^1) + a(k^1)c(-k^1)\big], \quad (13)$$

where the Klaiber's representation [9] of the massless vector current in terms of composite boson operators $c(k^1)$ was used (see Eq. (25) below). The Hamiltonian (13) is nondiagonal. Its diagonalization can be performed by a Bogoliubov transformation implemented by a unitary operator $U = \exp(iS)$ with

$$iS(\gamma) = \int_{-\infty}^{+\infty} dk^1\,\gamma(k^1)\big[c^\dagger(k^1)a^\dagger(-k^1) - \text{H.c.}\big]. \quad (14)$$

As a result, the real ground state has nontrivial structure, $|\tilde{\Omega}\rangle = \exp(-iS(\gamma_d))|0\rangle$ [10] (cf. Eqs. (29)–(32)) below).

All this is true provided the starting Hamiltonian is the right one. However, this is not the case. The point is that we did not use our knowledge of the operator solution (4) in the course of the derivation of the Hamiltonian (13). The solution tells us that in contradiction to the assumption of canonical quantization, the interacting field $\Psi(x)$ does not reduce to the free field $\psi(x)$ at $t=0$. In the case of models which are not exactly solvable, one knows $\gamma^\mu\partial_\mu\Psi(x)$ from the corresponding Dirac equation. This expression should not be used in the Lagrangian since the latter would vanish (extremum of the action). In our case, we actually know more, namely $\partial_\mu\Psi(x)$ from the full solution (4) which links $\Psi(x)$ to the free fields and this information *should* be used in the starting Lagrangian (1). This is similar to an elimination of a nondynamical field by using its constraint in a Lagrangian. Thus, the correct procedure is to insert the solution $\Psi(x)$ into $\mathcal{L}$. As already indicated, we should work with the point-split regularized Lagrangian. The hermiticity dictates that for field bilinears, we should take the combination $1/2(\overline{\psi}(x+\frac{\epsilon}{2})\psi(x-\frac{\epsilon}{2}) + \overline{\psi}(x-\frac{\epsilon}{2})\psi(x+\frac{\epsilon}{2}))$, etc. We have already seen that the hermitian definition of the currents led to a straightforward cancellation of the singular terms. For the scalar field, this similarly implies an automatic normal ordering of the kinetic term and of the mass term, since the singular parts cancel separately in these terms. The same thing occurs for the fermion mass term after inserting the regularized solution (8). From the fermion kinetic term, one generates in this way a normal-ordered *free* kinetic term *plus* the term canceling precisely the interacting term (after appropriately normal-ordering the latter [12] in quantum theory – the same cancellation occurs however also on the classical level). The net result is the normal-ordered free-field Lagrangian with the renormalized boson mass according to (10). The details will be given in a separate publication [8]. In this way, we arrive at $H = H_{0F} + H_{0B}$, where H_{0F} corresponds to the free massive fermion field and H_{0B} to the free scalar field with the mass $\tilde{\mu}$. Although the full Hamiltonian is the sum of *free* Hamiltonians and hence the ground state of the DCM is just the Fock vacuum, the model is not completely trivial: one generates the correct Heisenberg equations $i\partial_0\Psi(x) = -[H,\Psi(x)] = -i\alpha^1\partial_1\Psi + m\gamma^0\Psi - g\partial_0\phi\Psi - g\alpha^1\partial_1\phi\Psi$ with H. Correlation functions computed from the solution (4) are built from free fermion and boson two-point functions but depend on the coupling constant. Note also that in the conventional treatment the momentum operator contains interaction. This defect is cured in our approach.

The same picture is obtained in the LF analysis. Our notation is $x^\mu = (x^+, x^-)$, $\partial_\pm = \partial/\partial x^\pm$, $\hat{p}.x = 1/2(\hat{p}^-x^+ + p^+x^-)$, $\hat{p}^- = m^2/p^+$, where x^+, p^+ and $J^\mu = (J^+, J^-)$ are the LF time, momentum and current. The 2-dimensional free massive fermion field has two components, $\psi^\dagger(x) = ({\psi_1}^\dagger(x), {\psi_2}^\dagger(x))$, but no spinor structure, which is a welcome feature since there is no spin (no rotations) in one space dimension. The dynamical component $\psi_2(x)$ is expanded as

$$\psi_2(x) = \frac{1}{\sqrt{4\pi}}\int_0^\infty dp^+\left[\hat{b}(p^+)e^{-i\hat{p}\cdot x} + \hat{d}^\dagger(p^+)e^{i\hat{p}\cdot x}\right]. \quad (15)$$

The Fock operators $\hat{b}(p^+)$ and $\hat{d}(p^+)$ are the LF analogs of the operators $b(p^1)$ and $d(p^1)$ in the SL expansion (5) and satisfy the anticommutation relations $\{\hat{b}(p^+), \hat{b}^\dagger(q^+)\} = \{\hat{d}(p^+), \hat{d}^\dagger(q^+)\} = \delta(p^+ - q^+)$. The upper component of the LF fermion field is a nondynamical quantity determined from the constraint $2i\partial_-\psi_1(x) = m\psi_2(x)$ as

$$\psi_1(x) = \frac{1}{\sqrt{4\pi}}\int_0^\infty dp^+\,\frac{m}{p^+}\big[\hat{b}(p^+)e^{-i\hat{p}\cdot x} - \hat{d}^\dagger(p^+)e^{i\hat{p}\cdot x}\big]. \quad (16)$$

Inserting now the solution (4) of the field equations

$$\begin{aligned} &2i\partial_+\Psi_2(x) = m\Psi_1(x) + 2g\partial_+\phi(x)\Psi_2(x), \\ &2i\partial_-\Psi_1(x) = m\Psi_2(x) + 2g\partial_-\phi(x)\Psi_1(x) \end{aligned} \quad (17)$$

into the LF Lagrangian

$$\begin{aligned}\mathcal{L}_{lf} = {}& 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2 + i{\Psi_2}^\dagger\overset{\leftrightarrow}{\partial}_+\Psi_2 + i{\Psi_1}^\dagger\overset{\leftrightarrow}{\partial}_-\Psi_1 \\ & - m\big({\Psi_2}^\dagger\Psi_1 + {\Psi_1}^\dagger\Psi_2\big) - g\partial_+\phi J^+ - g\partial_-\phi J^-, \end{aligned} \quad (18)$$

we obtain the Lagrangian of the free LF massive fermion and boson fields with the corresponding LF Hamiltonian

$$P^- = \frac{1}{2}\int_{-\infty}^{+\infty} dx^-\left[m\big({\psi_2}^\dagger\psi_1 + {\psi_1}^\dagger\psi_2\big) + \frac{1}{2}\mu^2\phi^2\right]. \quad (19)$$

We recall that in the conventional treatment, one gets a controversial picture: the LF Hamiltonian still remains free while the SL Hamiltonian contains an interaction term (13) and its ground state is the coherent state $|\tilde{\Omega}\rangle$.

It is interesting that the analogous model with the axial vector current $J_5^\mu(x)$ replacing $J^\mu(x)$ is not solvable. One reason is that $J_5^\mu(x)$ is not conserved even classically, and hence the (pseudo)scalar field is not free. Moreover, the naive generalization of the solution (4) to $\Psi(x) = \exp\{-ig\gamma^5\phi(x)\}\psi(x)$ actually does not solve the corresponding Dirac equation due to $\{\gamma^\mu, \gamma^5\} = 0$. On the other hand, the Rothe–Stamatescu model [12] ($m=0$) is indeed exactly solvable but its structure is very simple, the Hamiltonian again corresponding to free fields. The interacting currents computed from the solution $\Psi(x)$ have a similar structure like the currents in (9) [8] without a need to introduce artificially an exponential of the line integral to maintain "gauge invariance" [12].

3. The Thirring model

The Thirring model [6] was extensively studied over a few decades as one of the prototype quantum field theories. It is a prominent example of the soluble models. A detailed analysis of its operator solution has been made in [9]. A systematic Hamiltonian treatment based on the model's solvability has, however, not been given so far.

The Lagrangian density of the massless Thirring model and the corresponding field equations read

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu}\overleftrightarrow{\partial_{\mu}}\Psi - \frac{1}{2}g J_{\mu}J^{\mu}, \tag{20}$$

$$i\gamma^{\mu}\partial_{\mu}\Psi(x) = gJ^{\mu}(x)\gamma_{\mu}\Psi(x), \tag{21}$$

As shown by Klaiber, a solution of the Dirac equation (21) is

$$\Psi(x) = e^{i(g/\sqrt{\pi})(\alpha j(x) - \beta\gamma^5\tilde{j}(x))}\psi(x),$$

$$\gamma^{\mu}\partial_{\mu}\psi(x) = 0. \tag{22}$$

The coefficients α and β satisfy $\alpha + \beta = 1$. The "potentials" $j(x)$ and $\tilde{j}(x)$ are connected to the free current $j^{\mu}(x)$ (taken as normal-ordered product of free fermion fields) according to $\partial_{\mu}j(x) = -\sqrt{\pi}j_{\mu}(x)$ and $\partial_{\mu}\tilde{j}(x) = \sqrt{\pi}\epsilon_{\mu\nu}j^{\nu}(x)$. This corresponds to replacing $J^{\mu}(x)$ by $j^{\mu}(x)$ in the field equation (21). The latter assumption is rather restrictive and does not represent the most general quantum solution, which can be obtained as follows. Setting for simplicity $\beta = 0$, consider the solution

$$\Psi(x) = e^{i(g/\sqrt{\pi})J(x)}\psi(x), \tag{23}$$

with the unknown potential $J(x)$ of the interacting current $J^{\mu}(x)$, i.e. defining $\partial_{\mu}J(x) = -\sqrt{\pi}J_{\mu}(x)$. Computing then the interacting current from the solution (23) using the point-splitting regularization as in Eq. (9), we arrive at $J^{\mu}(x) = {:}\overline{\psi}(x)\gamma^{\mu}\psi(x){:} + \frac{g}{2\pi}J^{\mu}(x)$. This relation tells us that the interacting current is simply the renormalized free current:

$$J^{\mu}(x) = G(g)j^{\mu}(x), \quad G(g) = \left(1 - \frac{g}{2\pi}\right)^{-1}. \tag{24}$$

We see that although the Klaiber's solution is qualitatively correct, the factor $G(g)$ was missed in [9]. This may have consequences for some aspects of bosonization of the massive Thirring model [13].

We then proceed by bosonization of the free vector current, using the Fock expansion of the massless spinor field, which is simply the $m = 0$ limit of Eq. (5). After the Fourier transformation, the current $j^{\mu}(x)$ is obtained in terms of boson operators $c(k^1)$:

$$j^{\mu}(x) = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{dk^1\,k^{\mu}}{\sqrt{2|k^1|}}\left\{c(k^1)e^{-i\hat{k}\cdot x} - c^{\dagger}(k^1)e^{i\hat{k}\cdot x}\right\},$$

$$c(k^1) = \frac{i}{\sqrt{|k^1|}}\int_{-\infty}^{+\infty}dp^1\,\{\theta(p^1k^1)[b^{\dagger}(p^1)b(p^1+k^1) - (b\to d)$$
$$+\,\epsilon(p^1)\theta(p^1(k^1-p^1))d(k^1-p^1)b(p^1)\}, \tag{25}$$

where $\epsilon(p^1)$ is the sign function. The composite operators c, $c^{\dagger}$ obey the canonical commutation relation,

$$[c(p^1), c^{\dagger}(q^1)] = \delta(p^1 - q^1), \quad c(k^1)|0\rangle = 0. \tag{26}$$

The Hamiltonian of the model is derived from the Lagrangian (20) after inserting the solution (22) into it. The contribution of the term $(i/2)\overline{\Psi}\gamma^{\mu}\overleftrightarrow{\partial_{\mu}}\Psi$ does not cancel the interaction term (indicating a less trivial dynamics than found in the DC model), it merely reverses its sign in comparison with the usual treatment [14,15]:

$$H = \int_{-\infty}^{+\infty}dx^1\left[-i\psi^{\dagger}\alpha^1\partial_1\psi - \frac{1}{2}g(J^0J^0 - J^1J^1)\right]. \tag{27}$$

In Fock representation, $H = H_0 + H_g$ has the form

$$H_0 = \int_{-\infty}^{+\infty}dp^1\,|p^1|[b^{\dagger}(p^1)b(p^1) + d^{\dagger}(p^1)d(p^1)], \tag{28}$$

$$H_g = G^2(g)\frac{g}{\pi}\int_{-\infty}^{+\infty}dk^1\,|k^1|[c^{\dagger}(k^1)c^{\dagger}(-k^1) + c(k^1)c(-k^1)].$$

H_g is not diagonal and thus $|0\rangle$ is not an eigenstate of the full Hamiltonian. To diagonalize H, we form the new Hamiltonians $\hat{H}_0 = H_0 - T$, $\hat{H}_g = H_g + T$ [16], where $T = \int_{-\infty}^{+\infty}dk^1\,|k^1|c^{\dagger}(k^1)c(k^1)$, and implement a Bogoliubov transformation by the unitary operator $U = e^{iS}$,

$$iS = \frac{1}{2}\int_{-\infty}^{+\infty}dk^1\,\gamma(k^1)[c^{\dagger}(k^1)c^{\dagger}(-k^1) - c(k^1)c(-k^1)],$$

with an unknown function $\gamma(k^1)$. $\hat{H}_0$ is invariant with respect to U. The operators $c(k^1)$ transform as

$$c(k^1) \to c(k^1)\cosh\gamma(k^1) - c^{\dagger}(-k^1)\sinh\gamma(k^1). \tag{29}$$

The new interaction Hamiltonian $e^{iS}\hat{H}_g e^{-iS}$ will be diagonal,

$$\hat{H}_g^d = \frac{1}{\cosh 2\gamma_d}\int_{-\infty}^{+\infty}dk^1\,|k^1|c^{\dagger}(k^1)c(k^1), \tag{30}$$

if $\gamma(k^1) = \gamma_d = \frac{1}{2}\operatorname{artanh}2G(g)\frac{g}{\pi}$. Then we have

$$e^{iS}(\hat{H}_0 + \hat{H}_g)e^{-iS}|0\rangle = 0 \tag{31}$$

and $|\Omega\rangle = e^{-iS}|0\rangle$ is the new vacuum state,

$$|\Omega\rangle = N\exp\left[-\kappa\int_{-\infty}^{+\infty}dp^1\,c^{\dagger}(p^1)c^{\dagger}(-p^1)\right]|0\rangle, \tag{32}$$

$\kappa = \frac{1}{2}\tanh\gamma_d$. $|\Omega\rangle$ is a coherent state of pairs of composite bosons with zero total momentum, $P^1|\Omega\rangle = 0$. The vacuum $|\Omega\rangle$ is invariant under axial $U(1)$ transformations

$$V(\beta)|\Omega\rangle = |\Omega\rangle, \quad V(\beta) = e^{i\beta Q_5},$$

$$Q_5 = \int_{-\infty}^{+\infty}dk^1\,\epsilon(k^1)[b^{\dagger}(k^1)b(k^1) - d^{\dagger}(k^1)d(k^1)]. \tag{33}$$

Thus, no chiral symmetry breaking occurs. This finding disagrees with the results [14] where a BCS type of ansatz for the vacuum state was used. However, the true vacuum has to be an eigenstate of the full Hamiltonian. Our $|\Omega\rangle$ is such a state while the BCS-like state is not.

Correlation functions have to be calculated using the vacuum $|\Omega\rangle$ and the solution (23) (regularized according to (8)), where

$$J(x) = \frac{G(g)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dq^1\,\frac{c(q^1)}{\sqrt{2|q^1|}}\theta(|q^1| - \eta)e^{-i\hat{q}.x} + \text{H.c.} \tag{34}$$

η is the conventional infrared cutoff. Additional details, including the solution (23) for arbitrary α and β, satisfying $\alpha + \beta = 1$, will be given elsewhere [8]. Our main result here is the generalization of the Klaiber's analysis to include the truly interacting vector current. Further, an explicit and non-approximative construction of the physical ground state of the Thirring model was performed. This has not been done before. The form of the vacuum state $|\Omega\rangle$ gives an indication of the structure of true ground states in more complicated relativistic models.

It would be very instructive to analyze the Thirring model also in the light-front version. Unfortunately, it is not known how to consistently quantize two-dimensional massless fermion fields in the LF formalism. The problem is that in one space dimension, one of the two components of the fermion field is a nondynamical variable satisfying a constraint which has to be inverted to express this component in terms of the dynamical one (see Eq. (16)). For the massless field, the right-hand side of the constraint vanishes and the corresponding information is missing in the theory. This problem is clearly visible for example in the work concerning the LF solution of the Schwinger model (see [17], e.g.) in which even a doubling of fermion degrees of freedom by quantizing the second component independently at the second characteristic $x^- = 0$ did not produce the usual physical picture of the model (nonvanishing fermion condensate, for example). It has to be admitted that the absence of a genuine LF solution of the models with massless two-dimensional fermion fields that are solvable in the SL form of the theory is puzzling and should be clarified.

4. The Federbush model

We will give here only a very brief discussion of our solution of the Federbush model. The main purpose is first to show that our modified canonical procedure removes the discrepancy between the structure of the SL and LF Hamiltonians also in the case of this model and second, to demonstrate advantages of the light-front formalism.

The Federbush model is defined by the Lagrangian

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial_\mu}\Psi - m\overline{\Psi}\Psi + \frac{i}{2}\overline{\Phi}\gamma^\mu \overleftrightarrow{\partial_\mu}\Phi - \mu\overline{\Phi}\Phi - g\epsilon_{\mu\nu}J^\mu H^\nu, \tag{35}$$

which describes two species of coupled fermion fields with masses m and μ [7]. Both currents $J^\mu = \overline{\Psi}\gamma^\mu\Psi$, $H^\mu = \overline{\Phi}\gamma^\mu\Phi$ are conserved. The coupled field equations

$$\begin{aligned} i\gamma^\mu\partial_\mu\Psi(x) &= m\Psi(x) + g\epsilon_{\mu\nu}\gamma^\mu H^\nu(x)\Psi(x),\\ i\gamma^\mu\partial_\mu\Phi(x) &= \mu\Phi(x) - g\epsilon_{\mu\nu}\gamma^\mu J^\nu(x)\Phi(x) \end{aligned} \tag{36}$$

are exactly solvable even for nonzero masses:

$$\begin{aligned} \Psi(x) &= e^{-i(g/\sqrt{\pi})h(x)}\psi(x), \quad i\gamma^\mu\partial_\mu\psi(x) = m\psi(x),\\ \Phi(x) &= e^{i(g/\sqrt{\pi})j(x)}\varphi(x), \quad i\gamma^\mu\partial_\mu\varphi(x) = \mu\varphi(x). \end{aligned} \tag{37}$$

In quantum theory, the above exponentials are usually regularized by the "triple-dot ordering" [18,19]. The potentials $j(x)$ and $h(x)$, given in terms of the free currents [20] as $\partial_\mu j(x) = \sqrt{\pi}\epsilon_{\mu\nu}j^\nu(x)$, $\partial_\mu h(x) = \sqrt{\pi}\epsilon_{\mu\nu}h^\nu(x)$ enter into the solutions (37) in an "off-diagonal" way. After inserting the solutions into the Lagrangian (35), the interaction term changes its sign yielding the Hamiltonian

$$H = H_0 + g\int_{-\infty}^{+\infty} dx^1\,\big(j^0h^1 - j^1h^0\big), \tag{38}$$

where H_0 is the sum of two free fermion Hamiltonians.

The LF field equations are also solved by (37) with the free LF fields $\psi(x)$, $\varphi(x)$; $j(x)$, $h(x)$ are given by $2\partial_- j(x) = \sqrt{\pi}j^+(x)$, $2\partial_- h(x) = \sqrt{\pi}h^+(x)$. In the standard LF treatment, one would simply insert the solution of the fermionic constraint into $\mathcal{L}$. This yields however the free LF Hamiltonian! It is only after inserting the full solution like in the SL case that one obtains the four-fermion interaction term also in the LF case:

$$P_g^- = \frac{1}{2}g\int_{-\infty}^{+\infty} dx^-\,\big(j^+h^- - j^-h^+\big). \tag{39}$$

The interacting SL Hamiltonian (38) contains terms composed solely from creation or annihilation operators, so the Fock vacuum is not its eigenstate. The diagonalization can be performed by a Bogoliubov transformation using a *massive* current bosonization. This is considerably more complicated than the massless case [9]. The massive analog (up to the kinematical factors, see below) of the boson operator $c(k^1)$ (25) is

$$\begin{aligned} A(k^1,t) = i\int_{-\infty}^{+\infty}\frac{dp^1}{\sqrt{E(k^1)}}\Big\{&\big[b^\dagger(p^1)b(k^1+p^1) - (b\to d)\big]\\ &\times \tilde f_1(p^1,p^1+k^1)e^{i(E(p^1)-E(k^1+p^1))t}\theta(k^1p^1)\\ &+\frac{1}{2}\big[b^\dagger(-p^1)b(k^1-p^1) - (b\to d)\big]\theta\big(p^1(k^1-p^1)\big)\\ &\times \tilde f_1(-p^1,k^1-p^1)e^{i(E(p^1)-E(k^1-p^1))t}\\ &+d(p^1)b(k^1-p^1)\epsilon(p^1)\theta\big(p^1(k^1-p^1)\big)\\ &\times \tilde f_2(p^1,k^1-p^1)e^{-i(E(p^1)+E(k^1-p^1))t}\\ &+d(p^1+k^1)b(-p^1)\theta(p^1k^1)\\ &\times \tilde f_2(-p^1,p^1+k^1)e^{-i(E(p^1)+E(k^1+p^1))t}\\ &-b(p^1)d\big(-(p^1-k^1)\big)\theta\big(k^1(p^1-k^1)\big)\\ &\times \tilde f_2\big(p^1,-(p^1-k^1)\big)e^{-i(E(p^1)+E(k^1-p^1))t}\Big\}. \end{aligned} \tag{40}$$

The quantities

$$\tilde f_i(p^1,q^1) = \frac{f_i(p^1,q^1)}{\sqrt{2E(p^1)}\sqrt{2E(q^1)}}, \quad i = 1,2 \tag{41}$$

with $f_1(p^1,q^1) = \sqrt{p^+q^+} + \sqrt{p^-q^-}$, $f_2(p^1,q^1) = \sqrt{p^+q^+} - \sqrt{p^-q^-}$ are two coefficient functions appearing in four spinor products of the form $u^\dagger(p^1)\gamma^0\gamma^\mu u(q^1)$ etc., which arise when one calculates the free vector current in the Fock representation from the expansion (5):

$$\begin{aligned} j^0(x) = \int_{-\infty}^{+\infty} d\tilde p^1\int_{-\infty}^{+\infty} d\tilde q^1\,\Big\{&\big[b^\dagger(p^1)b(q^1) - (b\to d)\big]\\ &\times e^{i(\hat p-\hat q)\dot x}f_1(p^1,q^1) + \big[b^\dagger(p^1)d^\dagger(q^1)e^{i(\hat p+\hat q)\dot x}\\ &+ d(q^1)b(p^1)e^{-i(\hat p+\hat q)\dot x}\big]f_2(p^1,q^1)\Big\}. \end{aligned} \tag{42}$$

For the component $j^1(x)$, the functions f_1 and f_2 are interchanged. They have the right kinematical structure to guarantee the correct transformation law for a vector current in two dimensions, $j'^\mu(x) = \Lambda^\mu{}_\nu j^\nu(x)$, where Λ is the matrix of the Lorentz transformations with the components $\Lambda^0{}_0 = \Lambda^1{}_1 = \cosh\omega$, $\Lambda^0{}_1 = \Lambda^1{}_0 = -\sinh\omega$, $\cosh\omega = (1 - v^2/c^2)^{-1/2}$. Since the operators $A(t,x^1)$ and $A^\dagger(t,k^1)$ (40) have been obtained by an inverse Fourier transformation from the assumed form of the current density

$$j^0(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk^1 \left[\tilde{A}(t, k^1) e^{ik^1 x^1} + \tilde{A}^\dagger(t, k^1) e^{-ik^1 x^1} \right] \tag{43}$$

after inserting the fermion representation (42) for $j^0(x)$, they automatically incorporate the correct transformation properties under Lorentz boosts. The massless bosonized current (25) with the operators $c(k^1)$, $c^\dagger(k^1)$ is obtained directly from (43) and (40) since the second, fourth and fifth term in (40) vanish for $m = 0$ due to vanishing of the corresponding f_i functions. In the massless limit, also the time evolution of the remaining two terms simplifies to the common factor $e^{-i|k^1|t}$ while their $\tilde{f}_i$ functions become the (irrelevant) factors $\theta(p^1(p^1 + k^1))$ and $\theta(p^1(p^1 - k^1))$. In the considered massive case these simplifications do not occur and the operators $A(k^1, t)$ are not Lorentz scalars since unlike the massless case it is not possible to extract a common factor $\hat{k}^\mu$ in their definition. There is also a separate time-evolution factor for each of the five terms and one cannot associate the operators $A(k^1, t)$ with a quantum of a composite scalar field like in the massless case. These operators are nevertheless a useful concept since their algebraic properties are simple at equal times and the Hamiltonian of the models becomes quadratic when expressed in terms of them [8]. The corresponding massive charge density in the bosonized form is then written as

$$j^0(x) = \frac{-i}{\sqrt{2}\pi} \int_{-\infty}^{+\infty} \frac{dk^1\, E(k^1)}{\sqrt{2E(k^1)}} A(k^1, t) e^{ik^1 x^1} + \text{H.c.} \tag{44}$$

On the other hand, the analogous LF operators $\hat{A}$, $\hat{A}^\dagger$ are much simpler and have a structure similar to the massless SL case (25):

$$\begin{aligned} \hat{A}(k^+, x^+) = i \int_0^{+\infty} \frac{dp^+}{\sqrt{k^+}} \{ & [\hat{b}^\dagger(p^+)\hat{b}(k^+ + p^+) - (\hat{b} \to d)] \\ & \times e^{\frac{i}{2}\frac{m^2 k^+ x^+}{p^+(k^+ + p^+)}} + \hat{d}(p^+)\hat{b}(k^+ - p^+) e^{-\frac{i}{2}\frac{m^2 k^+ x^+}{p^+(k^+ - p^+)}} \}, \end{aligned} \tag{45}$$

where

$$j^+(x) = \frac{-i}{2\pi} \int_0^\infty \frac{dk^+}{\sqrt{k^+}} k^+ \hat{A}(k^+, x^+) e^{-\frac{i}{2}k^+ x^-} + \text{H.c.} \tag{46}$$

In deriving $\hat{A}(k^+, x^+)$, we have used the Fock expansion (15). The field $\varphi_2(x)$ is expanded analogously. Due to $[\hat{A}(k^+), \hat{A}^\dagger(l^+)] = \delta(k^+ - l^+)$, valid at $x^+ = y^+$, the LF form of the solution (37) can be easily normal-ordered:

$$\begin{aligned} \Phi(x) &= Z(\epsilon) \exp\left\{ i\frac{g}{\sqrt{\pi}} \hat{A}^\dagger(x) \right\} \exp\left\{ i\frac{g}{\sqrt{\pi}} \hat{A}(x) \right\} \varphi(x), \\ \hat{A}(x) &= \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dk^+}{\sqrt{k^+}} \hat{A}(k^+, x^+) e^{-\frac{i}{2}k^+ x^-}. \end{aligned} \tag{47}$$

Similar formulae hold for the solution $\Psi(x)$ built from the operators $\hat{B}(k^+, x^+)$, $\hat{B}^\dagger(k^+, x^+)$ which are constructed from $h^+(x)$. The j^- and h^- currents contain the boson operators $\hat{C}(k^+, x^+)$, $\hat{D}(k^+, x^+)$ and their conjugates, related to $\hat{A}$, $\hat{A}^\dagger$, $\hat{B}$, $\hat{B}^\dagger$ via the current conservation. In contrast to its SL analog, the interacting LF Hamiltonian is diagonal and therefore $|0\rangle$ is its lowest-energy eigenstate:

$$\begin{aligned} P_g^- = \frac{g}{8\pi} \int_0^\infty dk^+ k^+ [& \hat{A}^\dagger(k^+)\hat{D}(k^+) + \hat{D}^\dagger(k^+)\hat{A}(k^+) \\ & - \hat{B}^\dagger(k^+)\hat{C}(k^+) - \hat{C}^\dagger(k^+)\hat{B}(k^+)]. \end{aligned} \tag{48}$$

Diagonalization of the bosonized SL Hamiltonian yielding the true SL vacuum state $|\Omega\rangle$ will be given in [8] together with a careful point-split regularized treatment.

The next step will be to compute the correlation functions in both schemes. This task is not simple since one needs to know the commutators of the composite boson operators at unequal times [8]. This is the place where complexities of the usual triple-dot ordering technique [19] enter into our bosonization approach. Irrespectively of this, the LF calculation will be much simpler: it works with Fock vacuum and simple operator structures while the SL formalism requires nontrivial coherent-state vacuum and complicated operator terms. The ultimate knowledge of exact correlation functions will allow us to get a deeper insight into the relation between the SL and LF forms of the Federbush model and of QFT in general.

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